Tentamen: Introduction to Plasma Physics

January 24, 2012

13.00-16.00 h

Please write clearly your name on *each* sheet, and on the first sheet also your student number, date of birth, and address. You can use either Dutch or English. This exam consists of four problems. A list of plasma formulas is attached.

PROBLEM 1 (25 points)

Consider a spatially nonuniform magnetostatic field expressed in terms of a Cartesian coordinate system by

$$\mathbf{B}(x,z) = B_0(1+\alpha x)\,\mathbf{e}_z\ , \qquad (1)$$

with B_0 and $\alpha \ll 1$ positive constants and $\mathbf{e}_z$ a unit vector in the z-direction.

a. Sketch the field lines of this magnetic field in the x-z plane and discuss qualitatively the trajectory of an electron moving in this field.

b. The magnetic field defined by equation (1) is not consistent with Maxwell's equations. We add a second component in the x-direction so that the total field becomes

$$\mathbf{B}(x,z) = B_0[\alpha z\,\mathbf{e}_x + (1+\alpha x)\,\mathbf{e}_z]\ . \qquad (2)$$

Show that this magnetic field is consistent with Maxwell's equations and again sketch the magnetic field lines in the x-z plane.

c. Determine the equation of the magnetic field lines.

d. Write down the Cartesian components of the equations of motion for an electron moving in the magnetic field given by equation (2).

e. Calculate the guiding-center drift velocity of an electron with parallel velocity $v_\parallel$ and perpendicular velocity $v_\perp$ moving close to the origin in the magnetic field given by equation (2).

PROBLEM 2 (25 points)

A tokamak is a toroidal magnetic plasma confinement device in which a toroidal magnetic field B_t is produced by external coils and a poloidal field B_p by inducing an electric toroidal current inside the plasma via transformer action. In the JET tokamak the toroidal magnetic field has a maximum value of $B_t(a) = 3.45$ T at the plasma boundary $r = a = 1.2$ m and is estimated to be 20% lower at the plasma center at $r = 0$. The total induced toroidal current is $I = 3.2$ MA and assumed to be uniformly distributed.

a. Using guiding-center theory discuss qualitatively why a poloidal magnetic field component is necessary for plasma confinement in a tokamak. Why is the toroidal magnetic field depressed at the plasma center $r = 0$?

b. In order to consider the equilibrium of the confined plasma we neglect toroidal effects and approximate the tokamak by a linear screw pinch with axial and azimuthal magnetic fields $B_z(r)$ and $B_\varphi(r)$, respectively. Show that the equilibrium pressure balance equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p + \frac{B_\varphi^2 + B_z^2}{2\mu_0}\right) = -\frac{B_\varphi^2}{\mu_0 r} \ . \qquad (3)$$

c. Calculate the azimuthal magnetic field $B_\varphi(r)$ inside the plasma.

d. Calculate the pressure distribution $p(r)$ inside the plasma assuming that the plasma pressure at the axis $r = 0$ is equal to $p(0)$.

e. Calculate the plasma pressure $p(0)$ at the center by using that $p(a) = 0$.

PROBLEM 3 (20 points)

Consider the propagation of righthand circularly polarized electromagnetic waves parallel to a constant magnetic field (i.e. R waves) through a uniform plasma consisting of mobile electrons and fixed ions. The dispersion relation of R waves is given by

$$\frac{c^2k^2}{\omega^2} = \frac{c^2}{v_{ph}^2} = 1 - \frac{\omega_p^2/\omega^2}{1-\omega_c/\omega} , \tag{4}$$

with c the velocity of light, k the wave number, ω the angular frequency and v_{ph} the phase velocity of the wave, ω_p the electron plasma frequency and ω_c (> 0) the electron cyclotron frequency.

a. Make a schematic drawing of the normalized phase velocity v_{ph}^2/c^2 as a function of the angular frequency ω. Calculate the cutoff frequency. Indicate in this diagram the positions of the cutoff and resonance frequencies and discuss what happens at these frequencies.

b. Whistlers are R waves with frequencies below the cyclotron frequency ω_c. Calculate at which frequency ω whistler waves have the highest phase velocity and the magnitude of this maximum phase velocity.

c. Calculate the group velocity $v_g = \mathrm{d}\omega/\mathrm{d}k$ of the whistler mode and show that $v_g \propto \omega^{1/2}$ for $\omega \ll \omega_c$. To show this you also have to use that for whistler waves $v_{ph}^2 \ll c^2$ or $\omega\omega_c/\omega_p^2 \ll 1$.

PROBLEM 4 (20 points)

Consider an axisymmetric cylindrical plasma in which the electric field and pressure gradients have only radial components, i.e. $\mathbf{E} = E\,\mathbf{e}_r$ and $\nabla p_i = \nabla p_e = \mathrm{d}p/\mathrm{d}r\,\mathbf{e}_r$, and the magnetic field only an axial component, i.e. $\mathbf{B} = B\,\mathbf{e}_z$. Neglecting the nonlinear $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term, the steady-state two-fluid equations can be written in the form

$$en(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i - e^2n^2\eta(\mathbf{u}_i - \mathbf{u}_e) = 0$$

$$-en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla p_e + e^2n^2\eta(\mathbf{u}_i - \mathbf{u}_e) = 0 ,$$

with e the unit of charge, n the plasma density, $\mathbf{u}_{i,e}$ the flow velocity of the electrons and ions and η the plasma resistivity.

a. From the θ components of these equations, show that $u_{ir} = u_{er}$. Explain qualitatively what this means.

b. From the r components, show that $u_{j\theta} = u_{E\times B} + u_{Dj}$ with $u_{E\times B}$ the $\mathbf{E}\times\mathbf{B}$ drift velocity and $u_{Dj} = \pm\nabla p_j/enB$ for ions and electrons, respectively.

c. Find an expression for u_{ir} showing that it does not depend on E_r.